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# Superstrings on $AdS_5 \times S^5$ supertwistor space

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Abstract: We derive the Green-Schwarz action on $AdS_5 \times S^5$ using an alternate version of the coset superspace construction. By Wick rotations and Lie algebra identifications we bring the coset to $\mathrm{GL}(4|4)/(\mathrm{Sp}(4) \otimes \mathrm{GL}(1))^2$, which allows us to represent the conformal transformations on unconstrained matrices. The derivation is more streamlined even for the bosonic sector, and conformal symmetry is manifest at every step. $\kappa$-symmetry gauge fixing is more transparent.

Keywords: Superstring Vacua, AdS-CFT Correspondance, Superspaces.

## Contents



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## 1. Introduction

The conjectured duality between $D = 4$, $\mathcal{N} = 4$ super Yang-Mills theory and type-IIB string theory compactified on five-dimensional anti-de-Sitter space $\times$ the five-sphere ($AdS_5 \times S^5$) [1] has created a considerable amount of interest in studying the corresponding string theory using world-sheet methods. Since the $AdS_5 \times S^5$ geometry is supported by the self-dual Ramond-Ramond (R-R) 5-form background, the standard NSR formalism does not apply in a straightforward way, while the manifestly space-time supersymmetric Green-Schwarz (GS) formalism seems more adequate.

In [2], based on maximal supersymmetry arguments, it has been shown that $AdS_5 \times S^5$, $AdS_7 \times S^4$ and $AdS_4 \times S^7$ are exact string and M-theory backgrounds. This raises the hope that in the case of $AdS_5 \times S^5$ one might find an exact solution of the corresponding conformal field theory. The first steps are, of course, writing down the world-sheet action and fixing its local symmetries.

The GS action in general supergravity background was written down in [3] for the type-IIA theory and in [4] for the type-IIB theory. These actions are written in terms of the space-time vielbein and a super two-form which have as lowest components the zehnbein and the Neveu-Schwarz-Neveu-Schwarz (NS-NS) two-form. The R-R field-strengths and the space-time spinors appear as higher components. For this reason these actions are not very practical, because the full solution (to all orders in superspace coordinates) of the supergravity constraints is not known.

An approach to constructing the GS action that circumvents this problem is the coset (super)space approach. This requires first that the bosonic background be a coset manifold, $G/H$, and furthermore that $G$ be the even part of a supergroup. For $AdS_5 \times S^5$ this approach was considered in [5] and extended to other $AdS_n \times S^n$ in [6]. In reference [5] the authors used the exponential parametrization for the coset elements and solved the Maurer-Cartan equations. The resulting action can be shown to possess $\kappa$-invariance. A slight disadvantage of this construction is that the auxiliary integral in the Wess-Zumino term cannot be performed explicitly without $\kappa$-gauge fixing.

Yet a third approach to constructing the GS action on $AdS_5 \times S^5$ was taken in [7]. This approach circumvents the 3-dimensional integral of the Wess-Zumino term by finding the $AdS_5 \times S^5$ potentials already in $\kappa$ gauge-fixed form. The resulting potentials are then used in an action of the type [4]. The resulting action was shown to be equivalent to the one obtained in the coset superspace approach [8].

The action resulting from the first two approaches has to be further $\kappa$-gauge fixed. This can be done in various ways [9]–[12], the result being a dramatic simplification of the action. The quantization is nevertheless still problematic since the action also contains quartic terms.

Here we use a version of the coset superspace approach. The main difference in our construction is that the space-time super-coordinates themselves are a representation of the superconformal group, not just a nonlinear realization. This implies that in the resulting action the superconformal symmetry is manifest. By Wick rotations and Lie-algebra identifications we reexpress the superconformal group as GL(4|4). We will therefore deal with unconstrained matrices. This eliminates the need of exponential parametrization of the coset representatives and dramatically simplifies the evaluation of the action. The proof of $\kappa$-invariance is immediate and requires the use of only a subset of the Maurer-Cartan equations. The flat space limit reveals that the coordinates are in the chiral representation of the supersymmetry algebra. Fixing $\kappa$-symmetry is more transparent.

In the following section we describe the manipulations needed to represent the superconformal transformations as unconstrained matrices and construct a new supercoset with $AdS_5 \times S_5$ as bosonic part. In the next section we describe two possible gauge fixings of the local symmetries of the coset construction which lead to bosonic sigma models with target spaces Wick-rotated forms of particular metrics on $AdS_5$ and $S^5$. In section 4 we complete the supercoset construction and prove that the action is $\kappa$-symmetric. Section 5 is devoted to constructing the flat space limit of our construction and showing by explicit computation that it is the standard GS action written in SO(5) $\otimes$ SO(5) spinor notation. Sections 6 and 7 are devoted to $\kappa$ gauge fixing. In section 6 we recover the Kallosh-Rahmfeld-Pesando gauge by requiring that the action be real upon undoing the Wick-rotations. In section 7 we relax the reality condition and construct a simpler action, which has only quadratic terms in fermions.

## 2. Coset construction

The coset superspace construction of [5] considers strings propagating on the superspace PSU$(2,2|4)/$SO$(4,1)\otimes$SO(5), which has as even part the $AdS_5\times S^5$ geometry. The exponential parametrization of the coset representatives is quite complicated and the explicit solution for the coset vielbein does not allow a transparent gauge fixing.

Starting from these observations we simplify the starting point of the construction. We perform Wick rotations and using the Lie-algebra identifications SO$(3,3)\simeq$ SL(4) and SO$(3,2)\simeq$ Sp(4) we obtain

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\otimes\mathrm{SO}(5)}\to\frac{\mathrm{PSL}(4|4)}{\mathrm{Sp}(4)\otimes\mathrm{Sp}(4)}\,. \tag{2.1}$$

PSL$(4|4)$, just as PSU$(2,2|4)$, is in fact already a coset; it is the coset of SL$(4|4)$ by the GL(1) group of elements with superdeterminant trivially equal to unity (matrices proportional to the identity). Since PSL$(4|4)$ does not have a representation in Mat$(4|4)$ but we would like to use a matrix representation, we further relax both the $P$ and the $S$ constraints by introducing additional scaling factors

$$\frac{\mathrm{PSL}(4|4)}{(\mathrm{Sp}(4))^2}\to\frac{\mathrm{GL}(4|4)}{(\mathrm{Sp}(4)\otimes\mathrm{GL}(1))^2}\,. \tag{2.2}$$

where the two GL(1)'s can be chosen to act separately on the upper-left and lower-right blocks. One may notice that only positive determinants are generated in this way. This slight shortcoming is fixed at the end, along with Wick rotating back. In this last formulation we have the advantage of using the unconstrained matrices of the general linear group and thus the exponential parametrization for the coset elements is no longer necessary. The coset elements ${Z_M}^A$ transform in the defining representation of the superconformal group, being therefore supertwistor-like objects. The index $M$ is acted upon by the superconformal group while the $(\mathrm{Sp}(4)\otimes\mathrm{GL}(1))^2$ acts on the $A$ indices.

These objects are not really supertwistors because the grading properties of GL$(4|4)$ are different from those of the usual supergroups. This prevents us from replacing GL$(4|4)$ spinors with "vectors" as

$$w^{MN}=Z^{AM}{Z_A}^N\,, \tag{2.3}$$

where ${Z_A}^N$ is the inverse of ${Z_N}^A$. However, we can still interpret the bosonic coordinates as bilinears in the inverse of the coset elements, in the usual way:

$$w^{mn} = Z^{am} {Z_a}^n \quad (AdS_5) \qquad w^{\bar{m}\bar{n}} = Z^{\bar{a}\bar{m}} {Z_{\bar{a}}}^{\bar{n}} \quad (S^5)\,. \tag{2.4}$$

The GL(1)⊗GL(1) factor in the coset can further be used to fix one degree of freedom for both $w^{mn}$ and $w^{\bar{m}\bar{n}}$ separately. This interpretation of $w^{mn}$ and $w^{\bar{m}\bar{n}}$ is indeed correct since any 5-vector can be written in this form as the following counting of components shows: $4\times4-(4\times5)/2-1=5$ where the second number is the dimension of Sp(4). In this language the superconformal symmetry of the background is not manifest any more; only the conformal and R symmetry are linearly realized. As will be shown later, this is nothing but a coordinate transformation from conformally flat metrics (in terms of $Z$) to other type of metrics. Another advantage of this construction is that it eliminates the use of gamma matrices and related identities, since the coordinates naturally appear as carrying spinor indices.

Similar "Wick-rotated" cosets exist also for $AdS_3 \times S^3$ and $AdS_2 \times S^2$. They are $\mathrm{GL}(2|2) \otimes \mathrm{GL}(2|2)/(\mathrm{Sp}(2) \otimes \mathrm{GL}(1) \otimes \mathrm{GL}(1))^2$ and $\mathrm{GL}(2|2)/(\mathrm{GL}(1))^4$, respectively. In the following we will concentrate on the $AdS_5 \times S^5$ construction.

## 3. Bosonic

The even part of the supercoset we constructed, which is $[\mathrm{GL}(4)/(\mathrm{Sp}(4) \otimes \mathrm{GL}(1))]^2$, should produce a sigma model with target space $AdS_5 \times S^5$. Since it is "a perfect square", we will discuss one of the two $\mathrm{GL}(4)/(\mathrm{Sp}(4) \otimes \mathrm{GL}(1))$ factors. The gauge fixing can be done in a variety of ways. There are, however, two extreme possibilities and these will lead to the standard conformally flat $AdS_5$ and $S^5$ metrics, respectively. For notational convenience we will break the GL(4) matrices into $2 \times 2$ blocks. The global, GL(4) indices will be split as $m \to (\mu, \mu')$ while the local Sp(4) indices will be split as $a \to (\alpha, \alpha')$. In general, the local Sp(4) transformations can be used to make diagonal blocks proportional to the identity matrix and to relate the off-diagonal blocks. The GL(1) transformations can further be used to pull out scales. The action for each of the two factors in the bosonic part of the coset is just

$$S = \int j^{\langle ab \rangle} \wedge * j_{\langle ab \rangle}\,, \tag{3.1}$$

where $j^{\langle ab \rangle}$ is the coset part of $j^{ab} = z^{am}\, d{z_m}^b$, $\mathbf{z} \in \mathrm{GL}(4)$, i.e. the antisymmetric $\Omega$-traceless part of $j^{ab}$. As a matter of notation, we will use bold-faced letters to denote a matrix as a whole as well as its associated vector. The antisymmetrization, tracelessness, index contraction and inverse of $\Omega$ are defined as

$$\begin{aligned} A_{[a}B_{b]} &= \frac{1}{2}(A_a B_b - A_b B_a) \qquad & A_{\langle a}B_{b\rangle} &= A_{[a}B_{b]} + \frac{1}{4}\Omega_{ab} A^c B_c \\ A^c B_c &= \Omega^{ab} A_b B_a \qquad & \Omega^{ab}\Omega_{ac} &= \delta^b_c\,. \end{aligned} \tag{3.2}$$

The $AdS_5$ sigma model in the upper half-space form emerges by picking a "triangular gauge", i.e. we use the Sp(4) invariance to set the coset representative in triangular form with the upper-left and lower-right $2 \times 2$ blocks proportional to the identity matrix. We further use the GL(1) invariance to scale to unity the upper-left block. Thus, the coset representative in $\mathrm{GL}(4)/\,\mathrm{Sp}(4) \otimes \mathrm{GL}(1)$ has the form:

$$z_m{}^a = \begin{pmatrix} \mathbb{1} & 0 \\ \mathbf{x} & x^0\,\mathbb{1} \end{pmatrix}. \tag{3.3}$$

The current, and its antisymmetric $\Omega$-traceless part, are given by:

$$j_a{}^b = \begin{pmatrix} 0 & 0 \\ \frac{d\mathbf{x}}{x^0} & \frac{dx^0}{x^0}\,\mathbb{1} \end{pmatrix} \qquad j^{\langle ab\rangle} = \frac{1}{2}\begin{pmatrix} -\frac{dx^0}{x^0}\,\omega & \frac{1}{x^0}\,d\mathbf{x}^T\,\omega \\ \frac{1}{x^0}\,\omega\,d\mathbf{x} & \frac{dx^0}{x^0}\,\omega \end{pmatrix},, \tag{3.4}$$

where we chose the Sp(4) metric to be $\Omega = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}\right)$. Using these expressions we immediately find that the action (3.1) gives the $AdS_5$ sigma model in the Wick-rotated Lobachevski upper half-space:

$$\mathcal{L}_{AdS_5} = \left(\frac{dx^0}{x^0}\right)^2 - \frac{dx^{\alpha'\beta}dx_{\alpha'\beta}}{2(x^0)^2} = \frac{(dx^0)^2 - (dx^1)^2 - (dx^3)^2 + (dx^2)^2 + (dx^4)^2}{(x^0)^2}. \tag{3.5}$$

Undoing the Wick rotation that took us from SO(4, 2) to SL(4) amounts to changing the sign of $(dx^3)^2$. If we further Wick-rotate to euclidean signature we obtain the standard Lobachevski upper half-space form of the metric.

The other extreme possibility is to first use the Sp(4) symmetry and set the coset representative in the antisymmetric form:

$$z_m{}^a = \begin{pmatrix} x_+\mathbb{1} & -\mathbf{x}^T \\ \mathbf{x} & x_-\mathbb{1} \end{pmatrix} \qquad \text{with } x_\pm = \frac{1}{2}(x^6 \pm x^0)\,. \tag{3.6}$$

This gauge produces, as we will see, the standard conformally flat metric on $S^5$, but works equally well for $AdS_5$. The antisymmetric traceless part of the current has the expression

$$j^{\langle ab\rangle} = \frac{1}{2\|\mathbf{z}\|^2}\begin{pmatrix} (x_+\,dx_- - x_-\,dx_+)\omega^{\alpha\beta} & x^{\beta'\alpha}\,dx^6 - x^6\,dx^{\beta'\alpha} \\ x^6\,dx^{\alpha'\beta} - x^{\alpha'\beta}\,dx^6 & (x_-\,dx_+ - x_+\,dx_-)\omega^{\alpha'\beta'} \end{pmatrix}, \tag{3.7}$$

with $\|\mathbf{z}\|^2 = x_+x_- - \frac{1}{2}x^{\alpha'\beta}x_{\alpha'\beta}$ and $x^{\alpha\beta'} = -x^{\beta'\alpha}$. This expression suggests that it is convenient to fix the GL(1) gauge by requiring $x^6 = R$. Defining the object $z = (x^0,\, x^{\alpha'\beta})$ and its square as $z^2 = -(x^0)^2 - \frac{1}{2}x^{\alpha'\beta}x_{\alpha'\beta}$, the current becomes:

$$j^{\langle ab\rangle} = \frac{R}{2(R^2+z^2)}\begin{pmatrix} -dx^0\,\omega^{\alpha\beta} & -dx^{\beta'\alpha} \\ dx^{\alpha'\beta} & dx^0\,\omega^{\alpha'\beta'} \end{pmatrix} \tag{3.8}$$

and the action (3.1) is the conformally flat sigma model:

$$\mathcal{L}_{S^5} = -j^{\langle\bar a\bar b\rangle}j_{\langle\bar a\bar b\rangle} = R^2\frac{(dz)^2}{(R^2+z^2)^2}\,. \tag{3.9}$$

In vector notation the square of the vector $z$ should be consistent with the Wick-rotations that take us from SO(5) to SO(2,3). Indeed, we find that

$$(dz)^2 = -(dx^0)^2 - (dx^1)^2 - (dx^3)^2 + (dx^2)^2 + (dx^4)^2 \,, \tag{3.10}$$

which has the needed $(2,3)$ signature.

One can arrive at the same result using only vector notation. In other words, one represents a 6-vector as a $4 \times 4$ matrix using the 5-dimensional $\gamma$ matrices and the charge conjugation matrix which can be chosen to be the Sp(4)-invariant metric:

$$z^{ab} = x_6 \Omega^{ab} + x^i (\Omega\gamma_i)^{ab} \qquad i = 1, \ldots, 5 \,. \tag{3.11}$$

It is then straightforward to compute the (antisymmetric traceless part of the) current which turns out to be:

$$j^{\langle ab \rangle} = \frac{1}{x_6^2 + x^i x^j \eta_{ij}} x^{[6} \, dx^{i]} \, (\Omega\gamma_i)^{ab} \,, \tag{3.12}$$

where $\eta$ is the SO(2,3)-invariant metric. As pointed out before, the GL(1) gauge freedom can be used to fix a component of $x = (x^i, x^6)$ to any nonvanishing value. Choosing again to fix the GL(1) gauge by requiring that $x^6 = R$, we obtain the $\sigma$-model in equation (3.9). (Relations between various coordinates are discussed in the appendix.)

## 4. Super

In the previous section we constructed two bosonic sigma models with target space the coset GL(4)/(Sp(4) $\otimes$ GL(1)). They can be used to build sigma models with target space the Wick-rotated $AdS_5 \times S^5$. Now we proceed with the supercoset construction started in section 2 and define the currents

$$J_A{}^B + A_A{}^B = Z_A{}^M \, dZ_M{}^B \,. \tag{4.1}$$

where $A_A{}^B$ is the $[\mathrm{Sp}(4) \otimes \mathrm{GL}(1)]^2$ connection and the bosonic parts of $J_A{}^B$ are analogous to the antisymmetric traceless part of $j_a{}^b$ from the previous section.

As shown in [13], if in a coset $G/H$ the subgroup $H$ is the invariant locus of a particular $\mathbb{Z}_4$ automorphism of the group $G$, then the extra integral in the Wess-Zumino term can be performed explicitly and the result expressed in terms of only components of the current (4.1). In other words, under the above assumptions the bosonic components of the NS-NS super two-form vanish while the rest are constant [14]. In our case $H = (\mathrm{Sp}(4) \otimes \mathrm{GL}(1))^2$, but only its $\mathrm{Sp}(4) \otimes \mathrm{Sp}(4)$ subgroup has the property mentioned above. This leads to slight deviations from the results of [13]. The action is given by:

$$S = \int_\Sigma J^{ab} \wedge *J_{ab} - J^{\bar{a}\bar{b}} \wedge *J_{\bar{a}\bar{b}} \pm \frac{1}{2} (E^{1/2} J^{a\bar{b}} \wedge J_{a\bar{b}} - E^{-1/2} J^{\bar{a}b} \wedge J_{\bar{a}b}) \,, \tag{4.2}$$

where $E = \mathrm{sdet}\, Z_M{}^A$. The relative coefficient is not fixed by the coset construction. However, the requirement of existence of $\kappa$ symmetry fixes its absolute value to be 1/2. In the gauge $E = 1$ our coset reduces to $\mathrm{PSL}(4|4)/\,\mathrm{Sp}(4) \otimes \mathrm{Sp}(4)$. However, as we will see in section 7, other choices of $E$ can be useful as well.

Showing that the action (4.2) is $\kappa$-symmetric and finding the corresponding transformations is not a complicated task. Following the model of construction of the GS superstring in general supergravity background we define the variations

$$\Delta_A{}^B = Z_A{}^M \delta Z_M{}^B \,. \tag{4.3}$$

The $\kappa$ symmetry transformations that we consider have a form similar to the standard ones. In particular, the bosonic variations vanish

$$\Delta_{ab} = 0 = \Delta_{\bar{a}\bar{b}} \,. \tag{4.4}$$

This can be understood by recalling that the generator for $\kappa$ transformations is $\not{p}\, d$ where $d = 0$ is the second-class constraint associated to the canonical momentum conjugate to the odd superspace coordinates and $p$ is the canonical momentum conjugate to $x$. By acting with it we get

$$\delta\, \cdot = [\not{p}\, d\,,\, \cdot\,] = [\not{p}\,,\, \cdot\,]d + \not{p}\,[d\,,\, \cdot\,]\,. \tag{4.5}$$

The first term vanishes by the second-class constraint while the second term contributes only fermionic variations.

The variations of the current $J$ can be obtained by splitting the naive variation of the right-hand-side of equation (4.1) in coset, Sp(4) and GL(1) parts and introducing the $(\mathrm{Sp}(4) \otimes \mathrm{GL}(1))^2$ covariant derivative $\mathcal{D}$ to absorb the Sp(4) and GL(1) pieces. With this provision it is then immediate to show that they are given by:

$$\begin{aligned}
\delta J_{ab} &= J_{\langle a}{}^{\bar{c}} \Delta_{\bar{c}|b\rangle} - \Delta_{\langle a}{}^{\bar{c}} J_{\bar{c}|b\rangle} \\
\delta J_{\bar{a}\bar{b}} &= J_{\langle \bar{a}}{}^{c} \Delta_{c|\bar{b}\rangle} - \Delta_{\langle \bar{a}}{}^{c} J_{c|\bar{b}\rangle} \\
\delta J_{a\bar{b}} &= \mathcal{D}\Delta_{a\bar{b}} + J_a{}^c \Delta_{c\bar{b}} - \Delta_a{}^{\bar{c}} J_{\bar{c}\bar{b}} \\
\delta J_{\bar{a}b} &= \mathcal{D}\Delta_{\bar{a}b} + \ J_{\bar{a}}{}^{\bar{c}} \Delta_{\bar{c}b} - \Delta_{\bar{a}}{}^{c} J_{cb}\,,
\end{aligned} \tag{4.6}$$

where we used the fact that $\delta E = E\, \mathrm{str}\, \Delta = \Delta_a{}^a - \Delta_{\bar{a}}{}^{\bar{a}} = 0$ and $\mathcal{D}E = 0$. In writing these variations we did not take into account the world sheet metric or, equivalently, the world sheet zweibein $e_i{}^{\mathcal{M}}$. Using light-cone coordinates we define its variation as

$$\delta e_\pm{}^{\mathcal{M}} = \Delta_{\pm+} e_-{}^{\mathcal{M}} + \Delta_{\pm-} e_+{}^{\mathcal{M}}\,, \tag{4.7}$$

which implies that the full variations of the current are

$$\delta_\kappa J_{\pm A}{}^B = \Delta_{\pm+} J_{-A}{}^B + \Delta_{\pm-} J_{+A}{}^B + \delta J_{\pm A}{}^B\,, \tag{4.8}$$

with the last term being given by (4.6).

With this starting point it is straightforward to determine the fermionic variations $\Delta_{a\bar{b}}$ and $\Delta_{\bar{a}b}$ as well as the variations of the zweibein, $\Delta_{\pm\pm}$. We need to use only two of the Maurer-Cartan equations

$$\begin{aligned} \mathcal{D}J_{a\bar{b}} + J_a{}^c \wedge J_{c\bar{b}} + J_a{}^{\bar{c}} \wedge J_{\bar{c}\bar{b}} &= 0 \\ \mathcal{D}J_{\bar{a}b} + J_{\bar{a}}{}^c \wedge J_{cb} + J_{\bar{a}}{}^{\bar{c}} \wedge J_{\bar{c}b} &= 0 \end{aligned} \tag{4.9}$$

together with the identities $J_{(i}{}^{ab}J_{j)ac} = \frac{1}{4}\delta^b_c J_{(i}{}^{ad}J_{j)ad}$ (and similarly for all barred indices) coming from the Sp(4) algebra. To make a long story short, the rest of the transformations that leave the action invariant are:

$$\begin{aligned} E^{1/4}\Delta_{a\bar{b}} \pm \sigma E^{-1/4}\Delta_{\bar{b}a} &= J_{\pm a}{}^c \kappa_{\mp c\bar{b}} + J_{\pm\bar{b}}{}^{\bar{c}}\kappa_{\mp a\bar{c}} \\ \Delta_{+-} + \Delta_{-+} &= 0 \\ \Delta_{\pm\pm} &= -\frac{1}{4}\left(E^{-1/4}J_{\pm}{}^{\bar{a}b} \mp E^{1/4}J_{\pm}{}^{b\bar{a}}\right)\kappa_{\pm b\bar{a}}\,, \end{aligned} \tag{4.10}$$

where $\sigma = \pm 1$ matches the sign of the Wess-Zumino term in the action (4.2) and the $\kappa$ parameters do not transform under one of the local GL(1) groups. As we have already pointed out before, this computation is much easier than the one performed in [5] because we do not need to decompose the currents in terms of coset generators.

## 5. Flat space limit

The flat space limit of the model we constructed is taken in two steps:

1. Add the identity to a coset element and, according to dimensional analysis, divide its fermionic and bosonic components by $\sqrt{R}$ and $R$, respectively, and

2. expand for $R \to \infty$.

The first step implies that a generic coset element is written as:

$$Z_M{}^A = \delta_M{}^A + \frac{1}{\sqrt{R}}f_M{}^A + \frac{1}{R}b_M{}^A\,. \tag{5.1}$$

Then, its inverse, as an expansion in $1/R$, is given by:

$$Z_B{}^N = \delta_B{}^N - \frac{1}{\sqrt{R}}\delta_B{}^M f_M{}^D\delta_D{}^N - \frac{1}{R}\left(\delta_B{}^M b_M{}^D\delta_D{}^N - \delta_B{}^M f_M{}^D\delta_D{}^P f_P{}^C\delta_C{}^N\right) + \cdots\,. \tag{5.2}$$

Using these relations the current (4.1) has the following expression:

$$\begin{aligned} J_A{}^B \approx \frac{1}{R}\Big[&\delta_A{}^M\left(db_M{}^B - f_M{}^D\delta_D{}^N df_N{}^B\right) + \sqrt{R}\delta_A{}^M\, df_M{}^B - \\ &- \frac{1}{\sqrt{R}}\delta_A{}^M\left(f_M{}^D\delta_D{}^N db_N{}^B + b_M{}^D\delta_D{}^N df_N{}^B - f_M{}^D\delta_D{}^P f_P{}^C\delta_C{}^N df_N{}^B\right)\Big] + \cdots\,. \end{aligned}$$

It is now immediate to separate the bosonic and fermionic components of the current. The even components of the current to leading order in $1/R$ are given by

$$\mathring{J}_a{}^b = \frac{1}{R}\delta_a{}^m \left(db_m{}^b - f_m{}^{\bar d}\delta_{\bar d}{}^{\bar n} df_{\bar n}{}^b\right), \tag{5.3}$$

which resembles the currents from the usual GS superstring. The expression for $\mathring{J}_{\bar a}{}^{\bar b}$ is obtained by replacing unbarred indices with barred indices and vice-versa in the above equation. The odd components are given by:

$$\mathring{J}_a{}^{\bar b} = \frac{1}{\sqrt{R}}\delta_a{}^m\, df_m{}^{\bar b} - \frac{1}{R^{3/2}}\delta_a{}^m \left(f_m{}^{\bar d}\delta_{\bar d}{}^{\bar n} db_{\bar n}{}^{\bar b} + b_m{}^d\delta_d{}^n df_n{}^{\bar b} - f_m{}^{\bar d}\delta_{\bar d}{}^{\bar p} f_{\bar p}{}^c\delta_c{}^n df_n{}^{\bar b}\right) \tag{5.4}$$

and similarly for $\mathring{J}_{\bar a}{}^b$. The current $J_{ab}$ appearing in the action (4.2) is actually only the traceless antisymmetric part of the r.h.s. of equation (5.3), as required by the coset construction. Before writing this explicitly, let us look for the field redefinitions that put the l.h.s. of the $\kappa$-transformations in a form similar to the usual one.

The $\kappa$ transformations in the flat space limit can be obtained from the equations (4.4) and (4.10) by making the rescaling $\kappa \to \sqrt{R}\kappa$, expanding around $R \to \infty$ and identifying the powers of $1/R$ on both sides of the equations. We get:

$$\delta b_{ab} - f_a{}^{\bar d}\,\delta f_{\bar d b} = 0 \qquad \delta b_{\bar a\bar b} - f_{\bar a}{}^d\,\delta f_{d\bar b} = 0\,, \tag{5.5}$$

from (4.4) while from (4.10) we get

$$\delta(f_{a\bar b} \pm f_{\bar b a}) = -R\,\mathring{J}_{\langle ac\rangle}\kappa_\pm{}^c{}_{\bar b} - R\,\mathring{J}_{\langle\bar b\bar c\rangle}\kappa_\pm{}_a{}^{\bar c}\,. \tag{5.6}$$

Since for the flat space GS superstring one usually writes $\delta\theta \sim \not{\Pi}\kappa$, the l.h.s. of transformations (5.6) suggest the following field redefinitions:

$$\theta_{a\bar b} = \frac{1}{2}(f_{a\bar b} + f_{\bar b a}) \qquad \theta_{\bar b a} = \frac{1}{2}(f_{a\bar b} - f_{\bar b a})\,. \tag{5.7}$$

In terms of these new objects the (traceless, antisymmetric part of the) bosonic currents become (we removed the overall factor of $1/R$):

$$\begin{aligned}
\mathring{J}_{ab} &= dy_{\langle ab\rangle} - (\theta_{\langle a}{}^{\bar d}\, d\theta_{b\rangle\bar d} + \theta_{\bar d\langle a}\, d\theta^{\bar d}{}_{b\rangle})\,; \qquad & y_{\langle ab\rangle} &\equiv b_{\langle ab\rangle} + \theta_{\langle a|}{}^{\bar d}\theta_{\bar d|b\rangle} \\
\mathring{J}_{\bar a\bar b} &= dy_{\langle\bar a\bar b\rangle} - (\theta_{\langle\bar a}{}^{d}\, d\theta_{\bar b\rangle d} + \theta_{d\langle\bar a}\, d\theta^{d}{}_{\bar b\rangle})\,; \qquad & y_{\langle\bar a\bar b\rangle} &\equiv b_{\langle\bar a\bar b\rangle} + \theta_{\langle\bar a|}{}^{d}\theta_{d|\bar b\rangle}\,.
\end{aligned} \tag{5.8}$$

For these currents to resemble the standard flat space ones, we would like to identify $y_{\langle ab\rangle}$ and $y_{\langle\bar a\bar b\rangle}$ with the space-time coordinates written in $SO(5)\otimes SO(5)$ spinor notation. As counted before, the number of components matches, but we also have to make sure that the rest of the action has the right form.

There are several contributions to the Wess-Zumino term. The leading one is $1/R\, df \wedge df$ but it is a total derivative and drops out. This is good, because the rest

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of the action is of the order $1/R^2$. The remaining contributions come from the cross terms in $\mathring{J}^{a\bar{b}}\mathring{J}_{a\bar{b}}$ and $\mathring{J}^{\bar{b}a}\mathring{J}_{\bar{b}a}$ together with a cross term between $1/R\, df \wedge df$ and a $1/R$ term coming from the expansion of $E^{\pm 1/2}$. In terms of the $\theta$ variables introduced above the Wess-Zumino term can be written, up to a total derivative, as

$$\begin{aligned}
\frac{1}{4}\left[E^{1/2}\mathring{J}^{\bar{b}a}\mathring{J}_{\bar{b}a} - E^{-1/2}\mathring{J}^{a\bar{b}}\mathring{J}_{a\bar{b}}\right] &= d(b^{[ab]} + \theta^{[a|\bar{d}}\theta_{\bar{d}}{}^{|b]}) \wedge (\theta_{[a}{}^{\bar{d}}\, d\theta_{b]\bar{d}} - \theta_{\bar{d}[a}\, d\theta^{\bar{d}}{}_{b]}) - \\
&- d(b^{[\bar{a}\bar{b}]} + \theta^{[\bar{a}|d}\theta_{d}{}^{|\bar{b}]}) \wedge (\theta_{[\bar{a}}{}^{d}\, d\theta_{\bar{b}]d} - \theta_{d[\bar{a}}\, d\theta^{d}{}_{\bar{b}]}) + \\
&+ \theta^{d[\bar{a}}\, d\theta_{d}{}^{\bar{b}]} \wedge \theta_{[\bar{a}}{}^{c}\, d\theta_{\bar{b}]c} + \theta^{\bar{d}[a}\, d\theta_{\bar{d}}{}^{b]} \wedge \theta_{[a}{}^{\bar{c}}\, d\theta_{b]\bar{c}} + \\
&+ d(b^{a}{}_{a} - b^{\bar{a}}{}_{\bar{a}} + 2\theta^{a\bar{b}}\theta_{\bar{b}a})(\theta^{c\bar{d}}\, d\theta_{c\bar{d}} + \theta^{\bar{d}c}\, d\theta_{\bar{d}c})\,. \qquad (5.9)
\end{aligned}$$

This is not the standard form of the Wess-Zumino term in flat space. However, we still need to separate the traces out of every antisymmetric factor in the equation above. The contributions from the first two lines completely cancel the fourth line while in the third line the traces cancel due to the antisymmetry of the $\wedge$-product. Thus, with the definition of $y$ from equation (5.8), the Wess-Zumino term becomes:

$$\begin{aligned}
\frac{1}{4}\left[E^{1/2}\mathring{J}^{a\bar{b}}\mathring{J}_{a\bar{b}} - E^{-1/2}\mathring{J}^{\bar{b}a}\mathring{J}_{\bar{b}a}\right] &= \theta^{d\langle\bar{a}}\, d\theta_{d}{}^{\bar{b}\rangle} \wedge \theta_{\langle\bar{a}}{}^{c}\, d\theta_{\bar{b}\rangle c} + \theta^{\bar{d}\langle a}\, d\theta_{\bar{d}}{}^{b\rangle} \wedge \theta_{\langle a}{}^{\bar{c}}\, d\theta_{b\rangle\bar{c}} + \\
&+ dy^{\langle ab\rangle} \wedge (\theta_{\langle a}{}^{\bar{d}}\, d\theta_{b\rangle\bar{d}} - \theta_{\bar{d}\langle a}\, d\theta^{\bar{d}}{}_{b\rangle}) - \\
&- dy^{\langle\bar{a}\bar{b}\rangle} \wedge (\theta_{\langle\bar{a}}{}^{d}\, d\theta_{\bar{b}\rangle d} - \theta_{d\langle\bar{a}}\, d\theta^{d}{}_{\bar{b}\rangle})\,. \qquad (5.10)
\end{aligned}$$

Combining this with the $\mathring{J}^{ab}\mathring{J}_{ab}$ and $\mathring{J}^{\bar{a}\bar{b}}\mathring{J}_{\bar{a}\bar{b}}$ terms constructed from equations (5.8) we get the usual Green-Schwarz action written in $\mathrm{SO}(5) \otimes \mathrm{SO}(5)$ spinor notation.

## 6. Kallosh-Rahmfeld-Pesando gauge

In the previous section we showed that the action (4.2) together with the $\kappa$ transformation rules (4.4) and (4.10) reproduce, in the flat space limit, the usual Green-Schwarz action. In this section we will find, for the curved space model, a $\kappa$-symmetry gauge that simplifies the action. Since the action also has an $(\mathrm{Sp}(4) \otimes \mathrm{GL}(1))^2$ local invariance, we need to fix it as well. In section 3 we constructed coset representatives of $\mathrm{GL}(4)/\,\mathrm{Sp}(4) \otimes \mathrm{GL}(1)$ that reproduce the $AdS_5$ and $S^5$ sigma models. With slight improvement they will continue to be a part of the supersymmetric construction.

There are many ways to parametrize the $\mathrm{GL}(4|4)/(\mathrm{Sp}(4) \otimes \mathrm{GL}(1))^2$ coset representatives. We will start with one that exhibits the $4+6$ splitting reminiscent of the D3 brane background. Schematically it looks as follows:

$$Z = [x^{(4)}][\theta][x^{(6)}]\,, \qquad (6.1)$$

where $[x^{(4)}]$ denotes the coordinates parallel to the brane while $[x^{(6)}]$ describes the coordinates orthogonal to it. An advantage of this parametrization is that it produces

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a separation of the transformation of the various components. As far as the even generators of the (4-dimensional) superconformal group are concerned, only the right-most factor transforms under the $R$-symmetry group $\mathrm{SL}(4) \subset \mathrm{SL}(4|4)$. Ordinary supersymmetry transformations mix the left-most factor with the middle one while the $S$-supersymmetry transformations mix all three factors together.

Using the Sp(4) gauges introduced in section 3 and noting that the coset representatives (3.3) giving the $AdS_5$ metric can be written as:

$$\mathcal{X} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathbf{x} & x_0\, \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathbf{x} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & x^0\, \mathbb{1} \end{pmatrix} \equiv \mathbf{X}\, \mathbf{X_0} \tag{6.2}$$

the explicit form of equation (6.1) is given by

$$Z = \begin{pmatrix} X_m{}^d & 0 \\ 0 & \delta_{\bar m}{}^{\bar n} \end{pmatrix} \begin{pmatrix} \delta_d{}^c & \theta_d{}^{\bar p} \\ \theta_{\bar n}{}^c & \delta_{\bar n}{}^{\bar p} \end{pmatrix} \begin{pmatrix} (X_0)_c{}^b & 0 \\ 0 & z_{\bar p}{}^{\bar b} \end{pmatrix}, \tag{6.3}$$

where we displayed the matrix indices to emphasize the transformation properties of various blocks. In the above expression $z_{\bar p \bar b}$ is, for the time being, an arbitrary $4 \times 4$ antisymmetric matrix representing an arbitrary 6-vector which will describe the $S^5$ part of the space. We will reduce to five its number of independent components by fixing the gauge for the last GL(1) factor. In section 3 we used the GL(1) transformations to set to 1 the sixth component of $\mathbf{z}$ and we obtained the conformally flat metric for the sphere. As will become apparent shortly, this is not a convenient gauge in the supersymmetric context. For this reason we choose to fix the GL(1) at the end. This will further simplify the action.

Once the Sp(4) gauges are fixed as above we are naturally led to pick a $\kappa$-symmetry gauge. In general one can set to zero any component of the spinors $\theta$ which is acted upon by the $\kappa$ transformations. If we do not want to further break the global $\mathrm{SL}(2) \otimes \mathrm{SL}(2)$ invariance surviving after Sp(4) gauge fixing, we are led to a fairly small number of choices:

$$\begin{array}{llllll} \theta_\alpha{}^{\bar n} = 0 & \text{and} & \theta_{\bar m}{}^{\alpha'} = 0\,; & \theta_{\alpha'}{}^{\bar n} = 0 & \text{and} & \theta_{\bar m}{}^{\alpha} = 0 \\ \theta_a{}^{\bar \mu} = 0 & \text{and} & \theta_{\bar \mu'}{}^{a} = 0\,; & \theta_a{}^{\bar \nu'} = 0 & \text{and} & \theta_{\bar \mu}{}^{a} = 0 \\ \theta_\alpha{}^{\bar n} = 0 & \text{and} & \theta_{\bar m}{}^{\alpha} = 0\,; & \theta_{\alpha'}{}^{\bar n} = 0 & \text{and} & \theta_{\bar m}{}^{\alpha'} = 0 \end{array} \tag{6.4}$$

and, of course, linear combinations thereof. To get a simple action one needs that the inverse of the fermion matrix, $[\theta]^{-1}$, has as few terms as possible. This narrows the possible choices to the first four listed above. A closer look at the structure of the current reveals that the first set of gauge conditions

$$\theta_\alpha{}^{\bar n} = 0 \qquad \text{and} \qquad \theta_{\bar m}{}^{\alpha'} = 0 \tag{6.5}$$

gives the simplest action. This is related to the fact that $[x^{(4)}]^{-1}\, d[x^{(4)}]$ has nonvanishing entries only in the block $(\alpha',\, \beta)$ and therefore has "destructive interference"



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with the fermion matrix. Furthermore, upon Wick-rotation back to $((4,1),(5,0))$ signature, the two parts of the gauge are complex conjugate to each other, which will lead to a real action. This will be the gauge that we will consider in the following.

At this point there exists the issue regarding the consistency of the gauge choice. By performing a $\kappa$ variation of the gauge conditions one can check that there is no left-over gauge invariance.

The currents can be easily computed. We have used the gauge conditions and the previous observations to cancel various terms as well as to write the result with an apparent $\mathrm{Sp}(4) \otimes \mathrm{Sp}(4)$ symmetry:

$$\begin{aligned} J_a{}^b &= (j_{AdS_5})_a{}^b - ({X_0}^{-1})_a{}^c \, \theta_c{}^{\bar m} \, d\theta_{\bar m}{}^d \, X_{0d}{}^b \\ J_{\bar a}{}^{\bar b} &= (z^{-1})_{\bar a}{}^{\bar m} \, dz_{\bar m}{}^{\bar b} = (j_{S^5})_{\bar a}{}^{\bar b} \\ J_a{}^{\bar b} &= ({X_0}^{-1})_a{}^c \, d\theta_c{}^{\bar m} \, z_{\bar m}{}^{\bar b} \\ J_{\bar a}{}^b &= z_{\bar a}{}^{\bar m} \, d\theta_{\bar m}{}^c \, (X_0)_d{}^b \, , \end{aligned} \tag{6.6}$$

where $(j_{AdS_5})_a{}^b$ and $(j_{S^5})_{\bar a}{}^{\bar b}$ are the $AdS_5$ and $S^5$ bosonic currents, respectively. It is now straightforward to write down the various terms in the action. The $(\bar a \bar b)$ part of the current is the same as with no fermions. As mentioned before, $\mathbf{z}$ describes a 6-vector with a scale invariance that remains to be fixed. However, we can already say that in the right coordinates $J_{\bar a}{}^{\bar b}$ produces the $S^5$ sigma model regardless of its norm. Indeed, as shown in the appendix, the coordinate transformation $z_m \to Y^{mn} = z^{am} z_a{}^n$ allows us to express the metric on $S^5$ in terms of *only* the unit vector pointing along $\mathbf{Y}$. Thus, regardless of how we chose to fix the norm of $\mathbf{Y}$, the $J^{\bar a \bar b} \, J_{\bar a \bar b}$ term in the action produces the standard metric on $S^5$. For this observation to be of any use we need the action to depend only on $Y$. As we will see shortly, this is indeed the case.

The $J^{ab} \, J_{ab}$ term in the action is equally easy. We get

$$J^{ab} \, J_{ab} = (j_{AdS_5})^{\alpha\beta} \, (j_{AdS_5})_{\alpha\beta} + (j_{AdS})^{\alpha'\beta'} \, (j_{AdS})_{\alpha'\beta'} + 2 J^{\alpha'\beta} \, J_{\alpha'\beta} \, , \tag{6.7}$$

where for the last term we used the antisymmetry of $J_{ab}$. The first two terms are equal and given by

$$(j_{AdS})^{\alpha\beta} \, (j_{AdS})_{\alpha\beta} = (j_{AdS})^{\alpha'\beta'} \, (j_{AdS})_{\alpha'\beta'} = \frac{1}{2} \left( \frac{dx^0}{x^0} \right)^2 \tag{6.8}$$

as follows immediately from equation (3.4). The last term, which also receives fermionic contributions, has the expression

$$2 J^{\alpha'\beta} \, J_{\alpha'\beta} = \frac{1}{2\,(x^0)^2} (dx^{\alpha'\beta} - \theta^{\alpha'\bar m} \, d\theta_{\bar m}{}^{\beta})(dx_{\alpha'\beta} - \theta_{\alpha'}{}^{\bar m} \, d\theta_{\bar m \beta}) \, . \tag{6.9}$$

As mentioned before, the structure of the Wess-Zumino term will decide whether the action can indeed be expressed only in terms of $\mathbf{Y}$. Using its definition, the first part of the Wess-Zumino term is given by

$$E^{-1/2} J^{\bar{a}b} J_{\bar{a}b} = \frac{1}{x^0\,|\mathbf{Y}|} Y^{\bar{m}\bar{n}}\, d\theta_{\bar{m}}{}^{\alpha} \wedge d\theta_{\bar{n}\alpha}\,, \tag{6.10}$$

while the second part takes the form

$$E^{1/2} J^{a\bar{b}} J_{a\bar{b}} = -\frac{|\mathbf{Y}|}{x^0} {Y^{-1}}_{\bar{m}\bar{n}}\, d\theta^{\alpha'\bar{m}} \wedge d\theta'_{\alpha}{}^{\bar{n}}\,. \tag{6.11}$$

The sign in the second equation comes from the fact that while $Y^{\bar{m}\bar{n}} = z^{\bar{a}\bar{m}} z_{\bar{a}}{}^{\bar{n}}$, its inverse is ${Y^{-1}}_{\bar{m}\bar{n}} = -z_{\bar{m}}{}^{\bar{a}} z_{\bar{n}\bar{a}}$ and $|\mathbf{Y}|^2 = \epsilon_{\bar{m}\bar{n}\bar{p}\bar{q}} Y^{\bar{p}\bar{q}}$. We get therefore that the action can be expressed in terms of only $\mathbf{Y}$ and its inverse. This in turn implies that we are free to choose its norm without altering the form of the action.

We argued before that with the gauge fixing considered here the action will be real upon Wick-rotating back to $((4,1),(5,0))$ signature and redefining $x$ as in equation (5.8). This might not be obvious from equations (6.10) and (6.11). To see this we first notice that using the (trivial) identity $\delta^i_{[a}\epsilon_{bcde]} = 0$, $\mathbf{Y}^{-1}$ is proportional to $\mathbf{Y}$:

$${Y^{-1}}_{\bar{m}\bar{n}} = -\frac{1}{2|\mathbf{Y}|^2} \epsilon_{\bar{m}\bar{n}\bar{p}\bar{q}} Y^{\bar{p}\bar{q}}\,. \tag{6.12}$$

Under these circumstances the Wess-Zumino term becomes real since using the SU(4) $\simeq$ SO(6) algebra it can be shown that after Wick rotation

$${Y^{\dagger}}_{\bar{m}\bar{n}} = \frac{1}{2} \epsilon_{\bar{m}\bar{n}\bar{p}\bar{q}} Y^{\bar{p}\bar{q}}\,, \tag{6.13}$$

where we used the fact that the norm of $\mathbf{Y}$ is real.

Recall that we are still to fix the last GL(1) gauge. From the equations (6.10)–(6.12) it is clear that the most useful gauge is

$$|\mathbf{Y}| = \frac{1}{x^0} \quad \Leftrightarrow \quad E = 1\,. \tag{6.14}$$

We therefore expect that the resulting action will be equivalent to the one obtained in [9]. This will indeed be the case.

As pointed out before, the Wess-Zumino term allows the action to be naturally written in terms of $\mathbf{Y}$. As shown in the appendix, the bosonic sigma model depends only on the unit vector along $\mathbf{Y}$. Thus, with the gauge (6.14) we get upon Wick-rotation the metric on the unit 5-sphere

$$J^{\bar{a}\bar{b}} J_{\bar{a}\bar{b}} = (d\Omega_5)^2\,, \tag{6.15}$$

which together with the $(j_{AdS})^{\alpha\beta}(j_{AdS})_{\alpha\beta} + (j_{AdS})^{\alpha'\beta'}(j_{AdS})_{\alpha'\beta'} = (dx^0/x^0)^2$ combine and give just

$$\frac{(d\mathbf{Y})^2}{\mathbf{Y}^2} \tag{6.16}$$

i.e. a conformally flat 6-dimensional space in cartesian coordinates.

Collecting various partial results from this section we find the $\kappa$ gauge fixed action to be

$$\mathcal{S} = \int_\Sigma \frac{1}{2\,(x^0)^2}\,\left(dx^{\alpha'\beta} + \frac{1}{2}\left[d\theta^{\alpha'\bar{m}}\,\theta_{\bar{m}}{}^{\beta} - \theta^{\alpha'\bar{m}}\,d\theta_{\bar{m}}{}^{\beta}\right]\right)^2 + \frac{1}{\mathbf{Y}^2}(d\mathbf{Y})\wedge *(d\mathbf{Y}) + \\ + \frac{1}{2}\left[\mathbf{Y}^{\bar{m}\bar{n}}\,d\theta_{\bar{m}}{}^{\alpha}\wedge\,d\theta_{\bar{n}\alpha} - \frac{1}{2}\epsilon_{\bar{m}\bar{n}\bar{p}\bar{q}}\mathbf{Y}^{\bar{m}\bar{n}}d\theta^{\alpha'\bar{p}}\wedge\,d\theta_{\alpha'}{}^{\bar{q}}\right]. \qquad (6.17)$$

This action has a form equivalent to the one derived in [10]. There and in [11] it has been checked that the fermionic quadratic form has no left-over zero modes around inhomogeneous ($\sigma$-dependent) string configurations. Furthermore, this gauge can be reached from any point in the space of such configurations.

## 7. Complex gauge

In deriving the equation (6.17) we have been guided by the requirement that the action be hermitean after we Wick-rotate back to the original coset superspace. If, however, we relax this assumption, we can find gauge conditions which bring the action to an even simpler form. Such an example is the gauge

$$\theta_{\bar{m}}{}^{n} = 0 \qquad (7.1)$$

together with the coset parametrization

$$Z = \begin{pmatrix} \delta_m{}^n & \theta_m{}^{\bar{n}} \\ \theta_{\bar{m}}{}^n & \delta_{\bar{m}}{}^{\bar{n}} \end{pmatrix}\begin{pmatrix} x_n{}^b & 0 \\ 0 & z_{\bar{p}}{}^{\bar{b}} \end{pmatrix}. \qquad (7.2)$$

For the time being we did not fix any of the local gauge invariances. These assumptions lead to the following set of currents:

$$\begin{aligned} J_a{}^b &= (x^{-1})_a{}^m\,dx_m{}^b = (j_{AdS})_a{}^b \\ J_{\bar{a}}{}^{\bar{b}} &= z_{\bar{a}}^{-1\bar{m}}\,dz_{\bar{m}}{}^{\bar{b}} = (j_{S^5})_{\bar{a}}{}^{\bar{b}} \\ J_a{}^{\bar{b}} &= x^{-1}{}_a{}^n\,d\theta_n{}^{\bar{m}}\,z_{\bar{m}}{}^{\bar{b}} \qquad J_{\bar{a}}{}^b = 0\,. \end{aligned} \qquad (7.3)$$

For later convenience we define, as in the previous section, $\mathbf{Y}$ and the corresponding object for $AdS_5$, say $\mathbf{W}$, as

$$Y^{\bar{m}\bar{n}} = z^{\bar{a}\bar{m}}z_{\bar{a}}{}^{\bar{n}} \qquad W^{mn} = x^{am}x_a{}^n \qquad (7.4)$$

in terms of which the superdeterminant is just $E = |\mathbf{Y}|^2/|\mathbf{W}|^2$. In terms of these objects the Wess-Zumino term is now:

$$E^{1/2}J^{a\bar{b}}J_{a\bar{b}} = -\frac{|\mathbf{Y}|}{|\mathbf{W}|}W^{np}d\theta_n{}^{\bar{m}}\wedge d\theta_p{}^{\bar{s}}\,Y^{-1}{}_{\bar{m}\bar{s}} = -\frac{1}{2|\mathbf{W}||\mathbf{Y}|}\epsilon_{\bar{r}\bar{t}\bar{m}\bar{s}}W^{np}Y^{\bar{r}\bar{t}}d\theta_n{}^{\bar{m}}\wedge d\theta_p{}^{\bar{n}}\,. \qquad (7.5)$$

We notice that without fixing any gauge, the Wess-Zumino term is expressed in terms of *only* the unit vectors pointing in the direction of $\mathbf{W}$ and $\mathbf{Y}$. This feature can be extended to the other terms as well. Choosing the Sp(4) gauges for both $x$ and $z$ as in equation (3.6), the $AdS_5$ and $S^5$ metrics are the scale invariant ones when expressed in terms of $\mathbf{W}$ and $\mathbf{Y}$, as shown in the appendix. We therefore write the action as

$$\mathcal{S} = \int_\Sigma |d\mathbf{W}|^2 - |d\mathbf{Y}|^2 + \frac{1}{4}\epsilon_{\bar{r}\bar{t}\bar{m}\bar{s}} W^{np} Y^{\bar{r}\bar{t}} d\theta_n{}^{\bar{m}} \wedge d\theta_p{}^{\bar{n}} \,, \tag{7.6}$$

with the provision that $\mathbf{W}$ and $\mathbf{Y}$ represent unit 6-vectors. Let us emphasize that this is not the result of $\mathrm{GL}(1)^2$ gauge fixing. As pointed out before, this action is not hermitean any more when Wick-rotated back to the original superspace. Its hermitean conjugate is the action obtained with the gauge fixing condition

$$\theta_m{}^{\bar{n}} = 0 \tag{7.7}$$

and the same parametrization of the coset elements as in (7.2).

## 8. Conclusions

In this paper we have followed a path different from the usual supercoset construction of the GS action on $AdS_5 \times S^5$. By Wick rotations and Lie algebra identifications we brought the coset to $\mathrm{GL}(4|4)/(\mathrm{Sp}(4) \otimes \mathrm{GL}(1))^2$. This modified starting point leads to a number of simplifications:

- unconstrained matrices are used instead of exponential parametrization of coset elements;
- spinor notation is natural and leads to the elimination of Dirac matrices and their identities;
- the derivation of the action is more streamlined;
- an easier proof of $\kappa$ invariance;
- the flat space limit can be taken explicitly, without the use of *a priori* knowledge of $AdS_5 \times S^5$ metric;
- $\kappa$ gauge fixing is more transparent; the Kallosh-Rahmfeld-Pesando gauge is easily obtained based on reality and conformal invariance requirements;
- the use of complex gauges is easier in our approach and it leads to simpler actions than previously considered.

In [15] it has been shown that the sigma model on the $\mathrm{PSL}(n|n)$ supergroup manifold is exactly conformal. Since the Sp(4) sigma models are also conformal, the GL(1)'s are abelian groups and our construction is GKO-like, we draw the conclusion that it is not unlikely that our construction leads to a conformal field theory.



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## A. Coordinate transformations

Using the parametrization of 6-vectors written in (3.11) it is easy to see that the interpretation of $w^{mn} = z^{am} {z_a}^n$ as space-time coordinates is, before GL(1) gauge fixing, nothing more that the coordinate transformation to a *manifestly* scale-invariant metric. This statement is actually independent of the dimension.

We start from an antisymmetric matrix $\mathbf{z}$ and define $\mathbf{w}$ as:

$$w^{ab} = z^{ca} \Omega_{cd} z^{db} \,. \tag{A.1}$$

Decomposing this equation in vector notation provides us with the relation between the D-vector $w^I$ and the D-vector $z^I$. It is:

$$w^0 = (z^0)^2 - (z^i)^2 \qquad w^i = 2\, z^0\, z^i \,, \tag{A.2}$$

where, as before, the indices $i$ and $j$ are $D-1$-dimensional indices.

Now starting from

$$ds^2 = \left( d \frac{w}{\sqrt{w^2}} \right)^2 = \frac{(dw^I)^2}{(w^I)^2} - \frac{(w^I\, dw_I)^2}{(w^I)^4} \tag{A.3}$$

and using the change of variables (A.2) we get the following line element:

$$ds^2 = 4 \frac{(z^i\, dz^0 - z^0\, dz^i)^2}{((z^0)^2 + (z^i)^2)^2} \tag{A.4}$$

i.e. a conformally flat metric in the gauge $z^0 = 1$.

This transformation is part of a one-parameter family of transformations which for particular values gives D-dimensional versions of the orthographic, stereographic and gnomonic projections of the complex plane.

One usually considers the projection of a sphere onto a plane tangent to the sphere. Here we implement this operation through a gauge condition. We start with the metric (A.3) and let the plane pass through the point $p = (w^0, 0, \ldots, 0)$ while the sphere has radius $R = \sqrt{(w^I)^2} = \sqrt{(w^0)^2 + (w^i)^2}$. Instead of considering a sphere of fixed radius and perform a projection from an arbitrary point $q$ on the line linking the center of the sphere and the point $p$, we let the sphere expand and require that the distance between $q$ and $p$ be fixed. Considering $q$ at distance $aR$ from the center of the sphere and noticing that the distance between the center of the sphere and $p$ is just $P = w^0$ we have the condition

$$P + aR = 1 + a \,. \tag{A.5}$$

For various values for $a$ one recovers well-known coordinate systems:

- $a = 0 \Rightarrow w^0 = 1$ and the metric becomes:

$$ds^2 = \frac{(dw^i)^2}{1+(w^i)^2} - \frac{(w^i\,dw_i)^2}{(1+(w^i)^2)^2} \tag{A.6}$$

i.e. gnomonic projection;

- $a = 1 \Rightarrow w^0 = 1 - \frac{1}{4}(w^i)^2$ and the metric becomes:

$$ds^2 = \frac{(dw^i)^2}{(1+\frac{1}{4}(w^i)^2)^2} \tag{A.7}$$

i.e. stereographic projection;

- $a = \infty \Rightarrow w^0 = \sqrt{1-(w^i)^2}$

$$ds^2 = (dw)^2 \qquad \text{with } (w^0)^2 + (w^i)^2 = 1 \tag{A.8}$$

i.e. orthographic projection.

Using the coordinate transformations (A.2) it is easy to translate these conditions in terms of $z$. The gauge condition $z^0 = 1$ together with the rescaling $z^i \to z^i/2$ reduces the equation (A.2) to the case $a = 1$.

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